Assignment # 9	Due on January 8th in class

1. In an inner product space M, suppose that there are $x, y \in M$, such that

$$|\langle x, y\rangle| = \|x\| \cdot \|y\| \ .$$

Prove that x and y are linear dependent. In other words, prove that there exists $\lambda_1, \lambda_2 \in \mathbb{C}$, such that $\lambda_1 x + \lambda_2 y = 0$ and at least one of λ_1 and λ_2 is not zero.

2. Let M be a Hilbert space, and let $\{e_1, e_2, \cdots\} \subset M$, satisfying $\|e_i\| = 1$, $\forall\, i \in \mathbb{N}_{\geq 1}$, and $\langle e_i, e_j\rangle = \delta_{ij}$, where δ is the Kronecker delta defined as

$$\delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} .$$

a) [**Bessel's Inequality**] Prove that for any $x \in M$, we have

$$\sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2 \leq \|x\|^2 .$$

Hint: Just need to show that for any $n \in \mathbb{N}_{\geq 1}$, $\sum_{i=1}^{n} |\langle x, e_i\rangle|^2 \leq \|x\|^2$.

b) Assume $\{e_1, e_2, \cdots\}^{\perp} = \{0\}$. Prove that for all $x \in M$, we have

$$\sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2 = \|x\|^2 .$$

Hint: Let $x_n = \sum_{i=1}^{n} \langle x, e_i \rangle e_i$. Show that $x_1, x_2, \cdots$ is a Cauchy sequence. As M is complete (because H is a Hilbert space), $y = \lim_{n\to\infty} x_n$ is in H. Easy to check that $\|y\|^2 = \lim_{n\to\infty} \|x_n\|^2 = \lim_{n\to\infty} \sum_{i=1}^{n} |\langle x, e_i \rangle|^2$. It only remains to show that $x = y$. In other words, $\lim_{n\to\infty} \|x - x_n\| = 0$.